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EQUAL TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS OF TWO OR LARGE NUMBER OF VARIABLES

The paper deals with the problem of transformations of a wide class of mathematical models, which are based on systems of differential equations, into a form that provides an effective computational implementation. The equivalence of the proposed transformations, which simplify the modelling process on multiprocessor computing platforms, is proved. The approach to effective numerical solution of differential equations on modern computer systems on the basis of the offered transformations has been shown.

Key words: Mathematical modelling of dynamic systems, Equivalent transformations of differential equations, iterative methods, computer implementation of mathematical models.

Target settings. The rapid development of computer-aided design technologies for complex dynamic systems stimulates the need to improve the methods of modelling them. Much of such systems are described by mathematical models based on systems of differential equations. Therefore, the scientific task of improving the technology of their modelling is actual for modern era of information technology development.

Actual scientific researches and issues analysis. The rapid development of computer-aided design technologies for complex dynamic systems stimulates the need to improve the methods of modelling them. Much of such systems are described by mathematical models based on systems of differential equations. Therefore, the scientific task of improving the technology of their modelling is actual for modern era of information technology development.

Actual scientific researches and issues analysis. One of the ways to improve computer modeling of dynamic systems is to increase the efficiency of computational implementation of mathematical models.

Uninvestigated parts of general matters defining. The problem of effective computer modelling of dynamical systems described by differential equations is one of the most important [1]. Most researchers are looking for its solution in adaptive discretization [2]. A number of works [3,4] are devoted to the problem of parallelization of calculations related to the solution of differential equations. Another part of the researchers [5,6] proposes solutions based on piecewise linear approximation. However, all of these decisions involve a loss of model accuracy**.**

The research objective. The purpose of this paper is increasing the efficiency of computer modeling of dynamic systems, which are described by differential equations, by converting them to a form convenient for computational implementation.

The statement of basic materials. Introduction. Consider identical transformations of differential equations of m-th order with one variable and m initial conditions. We introduce the notation $\frac{d^m y}{dt} = p^m y$. Requires any differential equation (hereinafter - DE) of m-th order to be able to represent in the form of an identical system of equations of the 1st order, which will be defined as a universal form (hereinafter - UF).

Any form of universal form for a differential equation of the m-th order, which contains derivatives of one variable (UFN) consists of m differential equations of the 1st order with m new variables ($y_i = \overline{1, m}$) and variable y, which remains. The form of UFN depends on the substitution equations, i.e. how the new variables relate to the initial variable y and its derivatives. Substitution equations always have the form of a system of linear equations.

Universal type 1 (derivative of one variable):

The differential equation has the form:

$$p^m y = f[p^{m-1}y, p^{m-2}y, \dots, py, y, t] \tag{1}$$

with initial conditions (hereinafter - IC):

$$p^i y(0) = C_{i+1}\big(i = 0, \overline{m-1}\big), \tag{2}$$

where m is the order of DE (maximum exponent of the derivative of DE).

Universal form 2 (derivatives of two variables):

The differential equation has the form:

$$p^m y = f[p^{m-1}y, p^{m-2}y, \dots, py, y, p^m x, p^{m-1}x, \dots, px, x, t] \tag{3}$$

with initial conditions:

$$\begin{cases} p^i y(0) = C_{i+1}(i = 0, \overline{m-1}) \\ p^i x(0) = D_{i+1}(i = 0, \overline{m-1}) \end{cases} \tag{4}$$

Universal form 3 (derivatives of three variables):

The differential equation has the form:

$$p^m y = \\ = f[p^{m-1}y, p^{m-2}y, \dots, py, y, p^m x, p^{m-1}x, \dots, px, x, p^m z, p^{m-1}z, \dots, pz, z, t] \tag{5}$$

with initial conditions:

$$\begin{cases} p^i y(0) = C_{i+1}(i = 0, \overline{m-1}) \\ p^i x(0) = D_{i+1}(i = 0, \overline{m-1}) \\ p^i z(0) = E_{i+1}(i = 0, \overline{m-1}) \end{cases} \tag{6}$$

Similarly, you can make UFN for DE with derivatives of N variables.

Universal form of the m-th order with 1 variable (UF1). The differential equation has the form:

$$p^m y = f[p^{m-1}y, p^{m-2}y, \dots, py, y, t]$$

with IC:

$$p^i y(0) = C_{i+1}\left(i = 0, \overline{m-1}\right),$$

where m is the order of DE (maximum exponent of the derivative of DE).

Any form of UF1 will contain a system with m DE's of the 1st order with m unknown and m initial conditions. The substitution equation reflects how the new variables $y_i(i = \overline{1,m})$ are related to the variable y and its derivatives. The linear form of DE(1) in the operator form corresponds to the fractional-rational transfer function (Fig. 1).

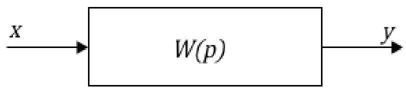

Fig. 1. Fractional-rational transfer function

$$y = W(p) * x, \text{ where:}$$

$$w(p) = \frac{1}{a_{m-1}p^{m-1} + a_{m-2}p^{m-2} + \ldots + a_1 p + a_0}$$

The simplest variant of the matrix of substitution equations for UV1 is a diagonal matrix:

$$\begin{cases} y = y_1 \\ py = y_2 \\ p^2 y = y_3 \\ \ldots \\ p^{m-1} y = y_m \end{cases}$$

The UF1 system thus obtained will be called the normal form and denoted as NF1.

Note that the m-1 differential equations of NF1's are obtained from the replacement equations, and the latter from the initial equation (1). Then NF1 will look like:

$$\begin{cases} y = y_1 \\ py_1 = y_2 \\ py_2 = y_3 \\ \ldots \\ py_{m-1} = y_m \\ py_m = [y_m, y_{m-1}, \ldots, y_2, \quad y_1, \quad t] \end{cases}$$

New initial conditions for NF1 are always obtained from the substitution equations. For NF1 new IC have the following form:

$$\begin{cases} y_1(0) = C_1 \\ y_2(0) = C_2 \\ y_3(0) = C_3 \\ \ldots \\ y_m(0) = C_m \end{cases}$$

The linear form of UF1 has the following form:

$$p^m y + a_{m-1}p^{m-1}y + \ldots + a_1 py + a_0 y = f(t)$$

The initial conditions remain unchanged. For this linear form UF1 NF1 will look like:

$$\begin{cases} y = y_1 \\ py_1 = y_2 \\ py_2 = y_3 \\ \ldots \\ py_{m-1} = y_m \\ py_m = -a_{m-1}y_m - a_{m-2}y_{m-1} - \ldots - a_1 y_2 - \\ -a_0 y_1 + f(t) \end{cases}$$

In addition to NF1 for UF1 can be used the so-called canonical form CF1 (hereinafter - CF1). CF1 is obtained provided that the matrix of coefficients of the substitution equation has a triangular shape. The substitution equation for CF1 will look like the following:

$$\begin{cases} y_1 = y \\ y_2 = py_1 + a_{m-1}y_1 = py + a_{m-1}y_1 \\ y_3 = py_2 + a_{m-2}y_1 = p^2 y + a_{m-1}py + a_{m-2}y \\ \ldots \\ y_m = p^{m-1}y + a_{m-1}p^{m-2}y + a_{m-2}p^{m-3}y + \ldots + \\ +a_1 y \end{cases}$$

Formulas for calculating the (m-1) DE’s are obtained from the substitution equation and the last equation is corrected so that the system is identical to the initial equation of the m-th degree.

The last equation of the system must be adjusted so that the resulting system of equations is identical to the initial equation of the m-th order (the adjusted function is denoted as F*).

Then the form CF1 will look like:

$$\begin{cases} y_1 = y \\ py_1 = y_2 + A_{m-1}y_1 \\ py_2 = y_3 + A_{m-2}y_1 \\ \ldots \\ py_{m-1} = y_m + A_1 y_1 \\ py_m = F^* + f(t) \end{cases}$$

For CF1 new IC have the following form:

$$\begin{cases} y_1(0) = C_1 \\ y_2(0) = C_2 + a_{m-1}C_1 \\ y_3(0) = C_3 + a_{m-1}C_2 + a_{m-2}C_1 \\ \ldots \\ y_m(0) = C_m + a_{m-1}C_{m-1} + a_{m-2}C_{m-2} + \ldots + a_1 C_1 \end{cases}$$

For the linear form UF1 (2), the adjusted function $\boldsymbol{F^*}$ **is equal to** $-\boldsymbol{A_0 y_1}$, i.e. $\boldsymbol{F^* = -A_0 y_1}$.

Other formulas of UF1 can also be used. The matrix of coefficients can be arbitrary. Such forms of UV1 are used for simultaneous modeling of several time functions using the method of solving the determining DE for modeling using operating units.

Universal form of the m-th order with 2 variable (UF2). Consider different variants of the universal form for the linear form of the differential equation UF2. We have a linear DE, which contains the derivatives of two variables (3) with the corresponding IC (4):

$$\begin{cases} p^i y(0) = C_{i+1}(i = 0, \overline{m-1}) \\ p^i x(0) = D_{i+1}(i = 0, \overline{m-1}) \end{cases}$$

The fractional-rational transfer function for this case has the form:

$$W(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + b_{m-2} p^{m-2} + \ldots + b_1 p + b_0}{a_{m-1} p^{m-1} + a_{m-2} p^{m-2} + \ldots + a_1 p + a_0}$$

According to the Lening-Bettin method, any subsequent form of UF is obtained from the previous one. Thus, from the form of UF1 is obtained the corresponding form of UF2. The essence of the Lening-Bettin method is that they proposed to add to each equation of NF1 a term proportional to the variable x.

The values of the coefficients are calculated so that the system of NF2 is identical to the initial equation of the m-th order, which has derivatives of 2 variables.

NF2 is obtained from NF1 by adding to each equation the terms $\alpha_i (i = \overline{0, m})$, then NF2 takes the form:

$$\begin{cases} y = y_1 + \alpha_m x \\ p y_1 = y_2 + \alpha_{m-1} x \\ p y_2 = y_3 + \alpha_{m-2} x \\ \ldots \\ p y_{m-1} = y_m + \alpha_1 x \\ p y_m = -a_{m-1} y_m - a_{m-2} y_{m-1} - \ldots - a_1 y_2 - \\ -a_0 y_1 + \alpha_0 x + f(t) \end{cases}$$

This corresponds to the following substitution equations:

$$\begin{cases} y_1 = y - \alpha_m x \\ y_2 = p y - \alpha_m p x - \alpha_{m-1} x \\ y_3 = p p^2 y - \alpha_m p^2 x - \alpha_{m-1} p x - \\ -\alpha_{m-2} x \\ \ldots \\ y_m = p^{m-1} y - \alpha_m p^{m-1} x - \alpha_{m-1} p^{m-2} x - \ldots - \\ -\alpha_2 p x - \alpha_1 x \end{cases}$$

New initial conditions for NF2:

$$\begin{cases} y_1(0) = C_1 - \alpha_m D_1 \\ y_2(0) = C_2 - \alpha_m D_2 - \alpha_{m-1} D_1 \\ y_3(0) = C_3 - \alpha_m D_3 - \alpha_{m-1} D_2 - \alpha_{m-2} D_1 \\ \dots \\ y_m(0) = C_m - \alpha_m D_m - \alpha_{m-1} D_{m-1} - \alpha_{m-2} D_{m-2} - \dots \\ -\alpha_1 D_1 \end{cases}$$

The system of equations for calculating $\alpha_i\left(i = \overline{0, m}\right)$ has the form:

$$\begin{cases} \alpha_m = b_m \\ \alpha_{m-1} + a_{m-1}\alpha_m = b_{m-1} \\ \alpha_{m-2} + a_{m-1}\alpha_{m-1} + a_{m-2}\alpha_m = b_{m-2} \\ \dots \\ \alpha_1 + a_{m-1}\alpha_2 + a_{m-2}\alpha_3 + \dots + a_1\alpha_m = b_1 \\ \alpha_0 + a_{m-1}\alpha_1 + a_{m-2}\alpha_2 + \dots + a_1\alpha_{m-1} + a_0\alpha_m = b_0 \end{cases}$$

According to the Lening-Bettin method, CF2 is also obtained by adding to each equation the terms $\alpha_i\left(i = \overline{0, m}\right)$, then it takes the form:

$$\begin{cases} y = y_1 + \alpha_m x \\ p y_1 = y_2 - a_{m-1} y_1 + \alpha_{m-1} x \\ p y_2 = y_3 - a_{m-2} y_1 + \alpha_{m-2} x \\ \dots \\ p y_{m-1} = y_m - a_1 y_1 + \alpha_1 x \\ p y_m = -a_0 y_1 + \alpha_0 x \end{cases}$$

The substitution equations for CF2 are:

$$\begin{cases} y_1 = y - \alpha_m x \\ y_2 = p y_1 + a_{m-1} y_1 - \alpha_{m-1} x = p y - \alpha_m p x + \\ +\alpha_{m-1}(y - \alpha_m x) - \alpha_{m-1} x \\ y_3 = p y_2 + a_{m-2} y_1 - \alpha_{m-2} x = p^2 y - \\ -\alpha_m p^2 x + a_{m-1}(p y - \alpha_m p x) - \alpha_{m-1} p x + \\ +a_{m-2}(y - \alpha_m x) - \alpha_{m-2} x \\ \dots \\ y_m = p y_{m-1} + a_1 y_1 - \alpha_1 x \end{cases}$$

New initial conditions for CF2:

$$\begin{cases} y_1(0) = C_1 - \alpha_m D_1 \\ y_2(0) = C_2 - \alpha_m D_2 + a_{m-1}(C_1 - \alpha_m D_1) - \alpha_{m-1} D_1 \\ y_3(0) = C_3 - \alpha_m D_3 + a_{m-1}(C_2 - \alpha_m D_2) - \alpha_{m-1} D_2 \\ +a_{m-2}(C_1 - \alpha_m D_1) - \alpha_{m-2} D_1 \\ \dots \end{cases}$$

Based on this, the system of equations for finding $\alpha_i x(i = \overline{0, m})$ has the form:

$$\begin{cases} \alpha_m = b_m \\ \alpha_{m-1} = b_{m-1} \\ \alpha_{m-2} = b_{m-2} \\ \dots \\ \alpha_1 = b_1 \\ \alpha_0 = b_0 \end{cases}$$

Conclusions. In this article the transformation of equations containing derivatives of more than 2 variables and the possibility of simultaneous modeling of several time functions was investigated and the ideology of Lening-Bettin method researched. The equivalence of the proposed transformations, which simplify the modelling process on multiprocessor computing platforms, is proved. Has been shown the approach to effective numerical solution of differential equations on modern computer systems on the basis of the offered transformations.

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